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A NOTE ON EULER'S (INCOMPLETE) Γ-FUNCTION

A note on Euler's (incomplete) Γ-function

by

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1. The subject of this section of this note was inspired by the following observation

Let

$$(1) \qquad f(t) = \sum_{n=1}^{N} a_n \cos(t\lambda_n), \quad (t \in R)$$

where all a's and λ's are real.

Clearly $f(t)$ is bounded and continuous so that we may consider the (one-sided) Laplace transform $\overset{\vee}{f}$ of f for $s > 0$:

$$(2) \qquad \overset{\vee}{f}(s) = \int_{0}^{\infty} e^{-st} f(t) \, dt =$$

$$= \sum_{n=1}^{N} a_n \int_{0}^{\infty} e^{-st} \cos(t\lambda_n) \, dt =$$

$$= \frac{1}{2} \sum_{n=1}^{N} a_n \int_{0}^{\infty} e^{-st}(e^{it\lambda_n} + e^{-it\lambda_n}) \, dt =$$

$$= \frac{1}{2} \sum_{n=1}^{N} a_n \left\{ \frac{1}{s - i\lambda_n} + \frac{1}{s + i\lambda_n} \right\}.$$

Differentiating k times with respect to s we obtain

$$(3) \qquad \int_{0}^{\infty} e^{-st} t^k f(t) \, dt = \frac{k!}{2} \sum_{n=1}^{N} a_n \left\{ \frac{1}{(s-i\lambda_n)^{k+1}} + \frac{1}{(s+i\lambda_n)^{k+1}} \right\},$$

or, equivalently

$$(4) \qquad \frac{1}{k!} \int_{0}^{\infty} e^{-u} u^k f(\tfrac{u}{s}) \, du = \frac{1}{2} \sum_{n=1}^{N} a_n \left\{ \frac{1}{\left(1- \frac{i\lambda_n}{s}\right)^{k+1}} + \frac{1}{\left(1+ \frac{i\lambda_n}{s}\right)^{k+1}} \right\}$$

Substituting $s = \frac{k}{t}$, $(t>0)$ in (4) and letting k tend to infinity we arrive at

$$(5) \qquad \lim_{k\to\infty} \int_{0}^{\infty} \frac{e^{-u} u^k}{k!} f\left(\tfrac{tu}{k}\right) \, du =$$

$$= \frac{1}{2} \sum_{n=1}^{N} a_n \left(e^{it\lambda_n} + e^{-it\lambda_n} \right) = f(t).$$

The main purpose of this section is to prove the validity of (5), or rather a generalization of it, for quite a large class of functions $f: R^+ \to \mathbb{C}$. In the meanwhile we will obtain some interesting results concerning Euler's (incomplete) Γ-function.

We start with stating the following:

THEOREM. *Let* $f: R^+ \to C$ *be such that*

(i) f *is integrable over every interval of the form* $(0,T)$ *where* $T > 0$,

(ii) *for some* $t_0 > 0$ *we have that the limits*

$$(6) \qquad \lim_{x \uparrow t_0} f(x) = L \text{ and } \lim_{x \downarrow t_0} f(x) = R$$

both exist and are finite,

(iii) *there exists a real constant* A *such that*

$$(7) \qquad f(x) = 0(e^{Ax}), \quad (x \to \infty).$$

Under these conditions we have

$$(8) \qquad \lim_{s \to \infty} \int_0^\infty \frac{e^{-u} u^s}{\Gamma(s+1)} \, f(\frac{t_0 u}{s}) \, du = \frac{L + R}{2}.$$

Before proving this theorem we will prove some lemmas.

LEMMA 1. *If* $0 < \alpha < 1$ *then*

$$(9) \qquad \lim_{s \to \infty} \int_0^{\alpha s} \frac{e^{-u} u^s}{\Gamma(s+1)} \, du = 0.$$

PROOF. For any fixed $s > 0$ the function $e^{-u} u^s$, $(u \in R^+)$, is increasing on the interval $(0,s)$ so that

$$(10) \qquad 0 < \int_0^{\alpha s} \frac{e^{-u} u^s}{\Gamma(s+1)} \, du < \alpha s \, \frac{e^{-\alpha s}(\alpha s)^s}{\Gamma(s+1)} =$$

$$= \alpha \; s \; \frac{e^{-\alpha s} \; \alpha^s \; s^s}{s^s e^{-s} \; \sqrt{2\pi s} \; e^{\mu(s)}} =$$

$$= \alpha\sqrt{s} \; \frac{(\alpha \; e^{1-\alpha})^s}{\sqrt{2\pi} \; e^{\mu(s)}}$$

where $\mu(s)$ is Binet's function which may be represented by

$$(11) \qquad \mu(s) = \int\limits_0^\infty \frac{e^{-st}}{t} \left\{ \frac{1}{e^t-1} - \frac{1}{t} + \frac{1}{2} \right\} dt, \; (s>0).$$

(See, for example, G. SANSONE & J. GERRETSEN, *Lectures on the theory of functions of a complex variable*, Noordhoff, Groningen, (1960) p. 216.)

Since

$$(12) \qquad 0 < \alpha \; e^{1-\alpha} < 1, \quad (0<\alpha<1)$$

and

$$(13) \qquad \lim_{s\to\infty} \mu(s) = 0$$

the lemma follows easily. □

LEMMA 2. *If* $\beta > 1$ *then*

$$(14) \qquad \lim_{s\to\infty} \int\limits_{\beta s}^\infty \frac{e^{-u} u^s}{\Gamma(s+1)} \; du = 0$$

PROOF. Observe that for $s > 0$ we have

$$(15) \qquad 0 < \int\limits_{\beta s}^\infty \frac{e^{-u} u^s}{\Gamma(s+1)} \; du = \int\limits_0^\infty \frac{e^{-x-\beta s}}{\Gamma(s+1)} \; (x+\beta s)^s \; dx =$$

$$= \frac{e^{-\beta s} \; \beta^s \; s^s}{s^s e^{-s} \; \sqrt{2\pi s} \; e^{\mu(s)}} \cdot \int\limits_0^\infty e^{-x}(1+ \frac{x}{\beta s})^s \; dx =$$

$$= \frac{(\beta \; e^{1-\beta})^s}{\sqrt{2\pi s} \; e^{\mu(s)}} \int\limits_0^\infty e^{-x}(1+ \frac{x}{\beta s})^s \; dx <$$

$$< \frac{(\beta e^{1-\beta})^s}{\sqrt{2\pi s} \; e^{\mu(s)}} \int\limits_0^\infty e^{-x} e^{\frac{x}{\beta}} dx = \frac{(\beta e^{1-\beta})^s}{\sqrt{2\pi s} \; e^{\mu(s)}} \; \frac{1}{1 - \frac{1}{\beta}}$$

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Since for $\beta > 1$ we also have

(16) $\qquad 0 < \beta \, e^{1-\beta} < 1$

the lemma follows easily. $\square$

LEMMA 3. *The function*

(17) $\qquad \dfrac{1}{\Gamma(s+1)} \displaystyle\int_0^s e^{-u} u^s du, \quad (s \in \mathbb{R}^+)$

tends (increasingly) to $\dfrac{1}{2}$ *when* $s \to \infty$

PROOF. By the substitution $u = s - x\sqrt{s}$ we obtain

(18) $\qquad \dfrac{1}{\Gamma(s+1)} \displaystyle\int_0^s e^{-u} u^s du = \dfrac{1}{\Gamma(s+1)} \int_0^{\sqrt{s}} e^{-s+x\sqrt{s}} (s-x\sqrt{s})^s \sqrt{s} \; dx =$

$\qquad = \dfrac{e^{-\mu(s)}}{\sqrt{2\pi}} \displaystyle\int_0^{\sqrt{s}} \exp\left\{ x\sqrt{s} + s \log(1 - \dfrac{x}{\sqrt{s}}) \right\} dx =$

$\qquad = \dfrac{e^{-\mu(s)}}{\sqrt{2\pi}} \displaystyle\int_0^{\sqrt{s}} \exp\left\{ -\dfrac{x^2}{2} - \sum_{n=3}^{\infty} \dfrac{x^n}{n} \dfrac{1}{s^{\frac{n}{2}-1}} \right\} dx =$

Now observe that

(19) $\qquad \dfrac{1}{e^t - 1} - \dfrac{1}{t} + \dfrac{1}{2} > 0, \quad (t>0)$

so that, using (11), it follows that $\mu(s)$ is decreasing on $\mathbb{R}^+$. Consequently $e^{-\mu(s)}$ is increasing on $\mathbb{R}^+$.

Also note that

(20) $\qquad \displaystyle\sum_{n=3}^{\infty} \dfrac{x^n}{n} \dfrac{1}{s^{\frac{n}{2}-1}}, \quad (0<x<\sqrt{s}; \; s>0)$

is decreasing in s for any fixed x subject to the conditions stated in (20).

Consequently we have established the "increasing part" of the lemma.

Using Lebesgue's dominated convergence theorem it also follows from (18)

that

(21)
$$\lim_{s\to\infty} \frac{1}{\Gamma(s+1)} \int_0^s e^{-u} u^s du = \frac{1}{\sqrt{2\pi}} \int_0^\infty e^{-\frac{x^2}{2}} dx = \frac{1}{2},$$

completing the proof of the lemma. □

LEMMA 4. *The function*

(22)
$$\frac{1}{\Gamma(s+1)} \int_s^\infty e^{-u} u^s du, \quad (s\in\mathbb{R}^+)$$

tends (decreasingly) to $\frac{1}{2}$ *when* $s \to \infty$.

PROOF. Since

(23)
$$\int_0^s e^{-u} u^s du + \int_s^\infty e^{-u} u^s du = \int_0^\infty e^{-u} u^s du = \Gamma(s+1)$$

the lemma is a direct consequence of lemma 3. □

After these preparations we proceed by proving the theorem stated before:

PROOF. For any given $\varepsilon > 0$ determine δ such that

(24)
$$0 < \delta < t_0$$

(25)
$$|f(x) - L| < \varepsilon, \quad (t_0-\delta<x<t_0)$$

and

(26)
$$|f(x) - R| < \varepsilon, \quad (t_0<x<t_0+\delta).$$

Write

(27)
$$\alpha = \frac{t_0 - \delta}{t_0} \quad \text{and} \quad \beta = \frac{t_0 + \delta}{t_0}$$

so that $0 < \alpha < 1$ and $\beta > 1$.

Also write

$$(28) \qquad \int_0^\infty \frac{e^{-u} u^s}{\Gamma(s+1)} \, f\left(\frac{t_0 u}{s}\right) du = \left\{ \int_0^{\alpha s} + \int_{\alpha s}^s + \int_s^{\beta s} + \int_{\beta s}^\infty \right\} \frac{e^{-u} u^s}{\Gamma(s+1)} \, f\left(\frac{t_0 u}{s}\right) du$$

Step 1. We first show that

$$(29) \qquad \lim_{s\to\infty} \int_0^{\alpha s} \frac{e^{-u} u^s}{\Gamma(s+1)} \, f\left(\frac{t_0 u}{s}\right) du = 0.$$

In order to see this we observe that

$$(30) \qquad \left| \int_0^{\alpha s} \frac{e^{-u} u^s}{\Gamma(s+1)} \, f\left(\frac{t_0 u}{s}\right) du \right| \le$$

$$\le \frac{e^{-\alpha s} (\alpha s)^s}{\Gamma(s+1)} \int_0^{\alpha s} \left| f\left(\frac{t_0 u}{s}\right) \right| du =$$

$$= \frac{(\alpha e^{1-\alpha})^s}{\sqrt{2\pi s} \, e^{\mu(s)}} \, \frac{s}{t_0} \int_0^{\alpha t_0} |f(x)| \, dx$$

so that (29) follows easily.

Step 2. If $\alpha s < u < s$ then

$$(31) \qquad t_0 - \delta < \frac{t_0 u}{s} < t_0$$

so that

$$(32) \qquad \left| f\left(\frac{t_0 u}{s}\right) - L \right| < \epsilon.$$

Hence

$$(33) \qquad \left| \int_{\alpha s}^s \frac{e^{-u} u^s}{\Gamma(s+1)} \, f\left(\frac{t_0 u}{s}\right) du - \frac{1}{2} L \right| =$$

$$= \left| \int_{\alpha s}^{s} \frac{e^{-u} u^{s}}{\Gamma(s+1)} \left\{ f\left(\frac{t_0 u}{s}\right) - L \right\} du + L \int_{\alpha s}^{s} \frac{e^{-u} u^{s}}{\Gamma(s+1)} du - \frac{1}{2} L \right| \leqq$$

$$\leqq \varepsilon \int_{\alpha s}^{s} \frac{e^{-u} u^{s}}{\Gamma(s+1)} du + |L| \left| \int_{0}^{s} \frac{e^{-u} u^{s}}{\Gamma(s+1)} du - \frac{1}{2} \right| + |L| \int_{0}^{\alpha s} \frac{e^{-u} u^{s}}{\Gamma(s+1)} du,$$

from which it is clear that

$$(34) \qquad \limsup_{s \to \infty} \left| \int_{\alpha s}^{s} \frac{e^{-u} u^{s}}{\Gamma(s+1)} f\left(\frac{t_0 u}{s}\right) du - \frac{1}{2} L \right| \leqq \frac{\varepsilon}{2}.$$

Step 3. Similarly as in step 2 one may show that

$$(35) \qquad \limsup_{s \to \infty} \left| \int_{s}^{\beta s} \frac{e^{-u} u^{s}}{\Gamma(s+1)} f\left(\frac{t_0 u}{s}\right) du - \frac{1}{2} R \right| \leqq \frac{\varepsilon}{2}.$$

Step 4. We have

$$(36) \qquad \lim_{s \to \infty} \int_{\beta s}^{\infty} \frac{e^{-u} u^{s}}{\Gamma(s+1)} f\left(\frac{t_0 u}{s}\right) du = 0.$$

In order to see this we proceed as follows: Determine the constants K, A and x_0 such that

$$(37) \qquad x_0 > \beta\, t_0,$$

$$(38) \qquad x_0 > 2\, t_0$$

and

$$(39) \qquad |f(x)| \leqq K\, e^{Ax}, \quad (x > x_0).$$

Since

$$(40) \qquad \left| \int_{\beta}^{\frac{x_0}{t_0} s} \frac{e^{-u} u^{s}}{\Gamma(s+1)} f\left(\frac{t_0 u}{s}\right) du \right| \leqq$$

$$\leqq \frac{e^{-\beta s} (\beta s)^{s}}{\Gamma(s+1)} \frac{s}{t_0} \int_{\beta t_0}^{x_0} |f(x)|\, dx$$

it follows as before that

$$(41) \qquad \lim_{s \to \infty} \int_{\beta s}^{\frac{x_0}{t_0}s} \frac{e^{-u}u^s}{\Gamma(s+1)} \, f\!\left(\frac{t_0 u}{s}\right) du = 0.$$

Hence, it suffices to show that

$$(42) \qquad \lim_{s \to \infty} \int_{\frac{x_0}{t_0}s}^{\infty} \frac{e^{-u}u^s}{\Gamma(s+1)} \, f\!\left(\frac{t_0 u}{s}\right) du = 0.$$

If $u > \dfrac{x_0}{t_0}\, s$ then $\dfrac{t_0 u}{s} > x_0$ so that, if in addition $s \geqq 2At_0$, we have

$$(43) \qquad \left| \int_{\frac{x_0}{t_0}s}^{\infty} \frac{e^{-u}u^s}{\Gamma(s+1)} \, f\!\left(\frac{t_0 u}{s}\right) du \right| \leqq$$

$$\leqq K \int_{\frac{x_0}{t_0}s}^{\infty} \frac{e^{-u}u^s}{\Gamma(s+1)} \, e^{A\frac{t_0 u}{s}} \, du =$$

$$= \frac{K}{\Gamma(s+1)} \int_{\frac{x_0}{t_0}s}^{\infty} e^{-u\left(1 - \frac{At_0}{s}\right)} u^s \, du =$$

$$= \frac{K}{\Gamma(s+1)} \int_{\frac{x_0 s}{t_0}\left(1 - \frac{At_0}{s}\right)}^{\infty} e^{-x} \left(\frac{x}{1 - \frac{At_0}{s}}\right)^s \frac{dx}{1 - \frac{At_0}{s}} \leqq$$

$$\leqq \frac{K}{\left(1 - \frac{At_0}{s}\right)^{s+1}} \int_{\frac{x_0 s}{2t_0}}^{\infty} \frac{e^{-x}x^s}{\Gamma(s+1)} \, dx \, .$$

Since $\dfrac{x_0}{2t_0} > 1$ by (38) it follows from lemma 2 that (42) holds true, completing the proof of the theorem. $\square$

REMARK. From lemma 4 one may derive very simply that the sequence

$$(44) \qquad \left\{ e^{-n} \sum_{k=0}^{n} \frac{n^k}{k!} \right\}_{n=0}^{\infty}$$

tends *decreasingly* to its limit $(=\tfrac{1}{2})$.

In order to see this we note that for any non negative integer n we have

$$(45) \qquad \frac{1}{\Gamma(n+1)} \int_{n}^{\infty} e^{-u} u^n \, du = \frac{1}{n!} \int_{0}^{\infty} e^{-(n+x)}(n+x)^n \, dx =$$

$$= \frac{1}{n!} \int_{0}^{\infty} e^{-n-x} \sum_{k=0}^{n} \binom{n}{k} n^k x^{n-k} \, dx =$$

$$= \frac{e^{-n}}{n!} \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} n^k \int_{0}^{\infty} e^{-x} x^{n-k} \, dx =$$

$$= e^{-n} \sum_{k=0}^{n} \frac{n^k}{k!} ,$$

so that our assertion follows from lemma 4.

2. The subject of this section of this note was inspired by the following observation.

For any positive integer n one has

$$(46) \qquad \log \left(\frac{n^n}{n!} \right)^{\frac{1}{n}} = \frac{1}{n} \log \left(\frac{n}{1} \frac{n}{2} \frac{n}{3} \cdots \frac{n}{n} \right) =$$

$$= -\frac{1}{n} \log \left(\frac{1}{n} \frac{2}{n} \frac{3}{n} \cdots \frac{n}{n} \right) =$$

$$= -\frac{1}{n} \sum_{k=0}^{n-1} \log \frac{n-k}{n} = -\frac{1}{n} \sum_{k=0}^{n-1} \log \left(1 - \frac{k}{n} \right) =$$

$$= \frac{1}{n} \sum_{k=0}^{n-1} \sum_{m=1}^{\infty} \frac{1}{m} \left(\frac{k}{n}\right)^m =$$

$$= \sum_{m=1}^{\infty} \frac{1}{m} \left\{ \frac{1}{n} \sum_{k=0}^{n-1} \left(\frac{k}{n}\right)^m \right\} =$$

$$= \sum_{m=1}^{\infty} \frac{1}{m} L_n(m),$$

where $L_n(m)$ denotes the n-th canonical lower Riemann sum corresponding to the function x^m, $0 \leq x \leq 1$.

It may be shown (see the author's Mathematical Centre Report ZW 39/75) that for any fixed m > 0, $L_n(m)$ is increasing in n. From this fact and (46) it then follows that the sequence

$$(47) \qquad \left\{ \left(\frac{n^n}{n!}\right)^{\frac{1}{n}} \right\}_{n=1}^{\infty}$$

is increasing.

As a generalization we prove the following

PROPOSITION. *The function* $f: \mathbb{R}^+ \to \mathbb{R}^+$ *defined by*

$$(48) \qquad f(s) = \left\{ \frac{s^s}{\Gamma(s+1)} \right\}^{\frac{1}{s}}, \quad (s>0)$$

is increasing on $\mathbb{R}^+$.

PROOF. For s > 0 we have

$$(49) \qquad \Gamma(s+1) = s^s e^{-s} \sqrt{2\pi s} \, e^{\mu(s)}$$

where $\mu(s)$ may be written as in (11). Since f(s) > 0 for s > 0 we may just as well prove that log f(s) is increasing on $\mathbb{R}^+$.

Observing that

$$(50) \qquad \log f(s) = 1 - \frac{\mu(s) + \frac{1}{2} \log (2\pi s)}{s}, \quad (s>0)$$

it is clearly sufficient to show that

(51) $\dfrac{d}{ds} \dfrac{\mu(s) + \frac{1}{2}\log(2\pi s)}{s} < 0, \quad (s>0).$

First we compute the derivative in (51):

(52) $\dfrac{d}{ds} \dfrac{\mu(s) + \frac{1}{2}\log(2\pi s)}{s} = \dfrac{s\{\mu'(s) + \frac{1}{2s}\} - \{\mu(s) + \frac{1}{2}\log(2\pi s)\}}{s^2}.$

Writing

(53) $\phi(s) = s\{\mu'(s) + \dfrac{1}{2s}\} - \{\mu(s) + \dfrac{1}{2}\log(2\pi s)\}, \quad (s>0)$

we will prove that

(54) $\phi(s) < 0, \quad (s>0).$

We first prove that

(55) $\lim_{s\downarrow 0} \phi(s) = 0.$

In order to see this we first note that

(56) $e^{\mu(s)} \sqrt{2\pi s} = \dfrac{e^s \, \Gamma(s+1)}{s^s}$

so that

(57) $\lim_{s\downarrow 0} \{\mu(s) + \dfrac{1}{2}\log(2\pi s)\} = \lim_{s\downarrow 0} \log \dfrac{e^s \, \Gamma(s+1)}{s^s} = \log 1 = 0.$

Next we observe that from the integral representation of $\mu(s)$ it follows that

(58) $\mu'(s) = -\displaystyle\int_0^\infty e^{-st}\{\dfrac{1}{e^t-1} - \dfrac{1}{t} + \dfrac{1}{2}\} \, dt =$

$= -\dfrac{1}{2s} + \displaystyle\int_0^\infty e^{-st}\{\dfrac{1}{t} - \dfrac{1}{e^t-1}\} \, dt$

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so that

$$(59) \qquad \mu'(s) + \frac{1}{2s} = \int_0^\infty e^{-st}\left\{\frac{1}{t} - \frac{1}{e^t-1}\right\} dt.$$

Since

$$\lim_{t\to\infty} \left\{\frac{1}{t} - \frac{1}{e^t-1}\right\} = 0$$

it follows from the general theory of Laplace transforms that

$$(60) \qquad \lim_{s\downarrow 0} s\{\mu'(s) + \frac{1}{2s}\} = 0.$$

Combining (57) and (60) it follows that (55) holds.

In view of (55) we may prove (54) by showing that

$$(61) \qquad \phi'(s) < 0, \quad (s>0).$$

In order to see this we note that

$$(62) \qquad \phi'(s) = (\mu'(s) + \frac{1}{2s}) + s\frac{d}{ds}(\mu'(s) + \frac{1}{2s}) - (\mu'(s) + \frac{1}{2s}) =$$

$$= s\frac{d}{ds}(\mu'(s) + \frac{1}{2s})$$

so that, using (59),

$$(63) \qquad \phi'(s) = - s\int_0^\infty e^{-st}\{1 - \frac{t}{e^t-1}\} dt, \quad (s>0).$$

Since

$$1 - \frac{t}{e^t-1} > 0, \quad (t>0)$$

it follows from (63) that (61) holds true, completing the proof. $\square$

REMARK. From (46) and the fact that $L_n(m)$ tends increasingly to $\int_0^1 x^m dx = \frac{1}{m+1}$

we may obtain a very transparent alternative proof of the well-known fact that

$$(64) \qquad \lim_{n \to \infty} \left(\frac{n^n}{n!} \right)^{\frac{1}{n}} = e.$$

Indeed

$$(65) \qquad \lim_{n \to \infty} \left(\frac{n^n}{n!} \right)^{\frac{1}{n}} = \lim_{n \to \infty} \exp\{ \sum_{m=1}^{\infty} \frac{1}{m} L_n(m) \} =$$

(by uniform convergence in n of the series involved)

$$= \exp\{ \sum_{m=1}^{\infty} \frac{1}{m} (\lim_{n \to \infty} L_n(m)) \} =$$

$$= \exp\{ \sum_{m=1}^{\infty} \frac{1}{m(m+1)} \} = \exp(1) = e.$$